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THE E_6 STATE SUM INVARIANT OF LENS SPACES

KENTA OKAZAKI

ABSTRACT. In this paper, we calculate the values of the E_6 state sum invariant for the lens spaces $L(p, q)$. In particular, we show that the values of the invariant are determined by $p \bmod 12$ and $q \bmod (p, 12)$. As a corollary, we show that the E_6 state sum is a homotopy invariant for the oriented lens spaces.

1. INTRODUCTION

In [5], Turaev and Viro constructed a state sum invariant of 3-manifolds based on their triangulations, by using the $6j$ -symbols of representations of the quantum group $U_q(\mathfrak{sl}_2)$. Further, Ocneanu [2] generalized the construction to the case of other types of $6j$ -symbols, say, the $6j$ -symbols of subfactors. *The E_6 state sum invariant* is the state sum invariant constructed from the $6j$ -symbols of the E_6 subfactor, which we denote by Z . Suzuki and Wakui [4] calculated the E_6 state sum invariant for some of the lens spaces, where they used the representation of the mapping class group of a torus $SL(2, \mathbb{Z})$.

In this paper, we calculate the E_6 state sum invariant for all of the lens spaces, as follows. For integers m, n , we denote by (m, n) the great common divisor of m and n . We put $\zeta = \exp(\pi\sqrt{-1}/12)$ and $[n] = (\zeta^n - \zeta^{-n})/(\zeta - \zeta^{-1})$ for an integer n , noting that

$$\begin{aligned} [12 - n] &= [n], & [n + 12] &= -[n], \\ [2] &= (1 + \sqrt{3})/\sqrt{2}, & [3] &= 1 + \sqrt{3}, & [4] &= (3 + \sqrt{3})/\sqrt{2}. \end{aligned}$$

Theorem 1.1. *For coprime integers p and q , the E_6 state sum invariant of the lens space $L(p, q)$ is given as*

$$(1.1) \quad Z(L(p, q)) = \begin{cases} |[p]| & \text{if } (p, 12) = 1, \\ [4][3]/[2] & \text{if } (p, 12) = 2, 6, \\ \zeta^{\pm 3}[4] & \text{if } (p, 12) = 3 \text{ and } q \equiv \pm 1 \pmod{3}, \\ 2\zeta^{\pm 2}[3] & \text{if } (p, 12) = 4 \text{ and } q \equiv \pm 1 \pmod{4}, \\ 2[4][3]/[2] & \text{if } 12|p \text{ and } q \equiv \pm 1 \pmod{12}, \\ 0 & \text{if } 12|p \text{ and } q \equiv \pm 5 \pmod{12}. \end{cases}$$

In particular, the value of $Z(L(p, q))$ is determined by $p \pmod{12}$ and $q \pmod{(p, 12)}$.

We note that we normalize the invariant so that $Z(S^3) = 1$. Thus, our Z is equal to wZ in [4], where we put $w = 2 + [3]^2 = 6 + 2\sqrt{3}$.

Corollary 1.2. *If there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p, q)$ and $L(p', q')$, then $Z(L(p, q)) = Z(L(p', q'))$.*

We note that from Theorem 1.1 the E_6 state sum invariant distinguishes $L(p, q)$ from $L(p, -q)$ if and only if $(p, 12) = 3$ or 4 . This is a generalization of [4, Corollary 4.3].

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2. THE CALCULATION OF THE E_6 STATE SUM INVARIANT

In this section, we briefly review the calculation of the E_6 state sum invariant for the lens spaces. Suzuki and Wakui [4] defined the representation $\rho : SL(2, \mathbb{Z}) \rightarrow GL_{10}(\mathbb{C})$ by

$$\rho(S)$$

$$= \frac{1}{w} \begin{pmatrix} 1 & [3] & 1 & [2]^2 & [3] & [3] & \frac{[4][3]}{[2]} & [3] & [3] & [2]^2 \\ [3] & \frac{[4][3]}{[2]} \sqrt{-1} & -[3] & -[3] & 0 & -\frac{[4][3]}{[2]} \sqrt{-1} & 0 & [3] & -[3] & [3] \\ 1 & -[3] & 1 & [2]^2 & -[3] & -[3] & -\frac{[4][3]}{[2]} & [3] & [3] & [2]^2 \\ [2]^2 & -[3] & [2]^2 & 1 & -[3] & -[3] & \frac{[4][3]}{[2]} & -[3] & -[3] & 1 \\ [3] & 0 & -[3] & -[3] & 0 & 0 & 0 & -2[3] & 2[3] & [3] \\ [3] & -\frac{[4][3]}{[2]} \sqrt{-1} & -[3] & -[3] & 0 & \frac{[4][3]}{[2]} \sqrt{-1} & 0 & [3] & -[3] & [3] \\ \frac{[4][3]}{[2]} & 0 & -\frac{[4][3]}{[2]} & \frac{[4][3]}{[2]} & 0 & 0 & 0 & 0 & 0 & -\frac{[4][3]}{[2]} \\ [2] & [3] & [3] & -[3] & -2[3] & [3] & 0 & [3] & [3] & -[3] \\ [3] & -[3] & [3] & -[3] & 2[3] & -[3] & 0 & [3] & [3] & -[3] \\ [2]^2 & [3] & [2]^2 & 1 & [3] & [3] & -\frac{[4][3]}{[2]} & -[3] & -[3] & 1 \end{pmatrix},$$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\zeta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are generators of $SL(2, \mathbb{Z})$.

Let p, q be coprime integers. We choose a continued fraction expansion of p/q :

$$\frac{p}{q} = c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_n}}}.$$

From this, we have that

$$\begin{pmatrix} a & p \\ b & q \end{pmatrix} = T^{c_1} S T^{c_2} \cdots S T^{c_{n-1}} S T^{c_n} \in SL(2, \mathbb{Z})$$

for some integers a, b . From [4, Lemma 4.2], one can derive that the E_6 state sum invariant of lens spaces are given as

$$(2.1) \quad \begin{aligned} Z(L(p, q)) &= w^t \mathbf{e} \rho(ST^{c_1} S T^{c_2} \cdots S T^{c_{n-1}} S T^{c_n} S) \mathbf{e} \\ &= w^t \mathbf{e} \rho \left(\begin{pmatrix} -q & b \\ p & -a \end{pmatrix} \right) \mathbf{e}, \end{aligned}$$

where we put $\mathbf{e} = {}^t(1, 0, 0, 0, 0, 0, 0, 0, 0)$. We note that this value does not depend on the choice of continued fraction expansions of p/q (and that of a and b).

3. PROOF OF THE THEOREM

In this section, we prove Theorem 1.1 and Corollary 1.2. In order to show Theorem 1.1, we show Proposition 3.4, which says that the values of $Z(L(p, q))$ have period 12 for p and q . In order to show Proposition 3.4, we show Lemmas 3.1 and 3.3, as follows.

Lemma 3.1. *Let p, q, p', q' be integers satisfying $(p, q) = 1$, $(p', q') = 1$ and $p \equiv p', q \equiv q' \pmod{12}$. Then, there exist integers a, b, a', b' such that*

$$aq - bp = 1, a'q' - b'p' = 1 \quad \text{and} \quad a \equiv a', b \equiv b' \pmod{12}.$$

Proof. We put integers a and b satisfying $aq - bp = 1$. Further, we put

$$\begin{pmatrix} a' & p' \\ b' & q' \end{pmatrix} = \begin{pmatrix} a & p \\ b & q \end{pmatrix} + 12 \begin{pmatrix} x & z \\ y & w \end{pmatrix},$$

noting that, by assumption, z and w are determined uniquely. It is sufficient to show that there exist integers x and y satisfying $a'q' - b'p' = 1$. The determinant of the right-hand side of the above formula is equal to

$$\begin{aligned} (3.1) \quad & (a + 12x)(q + 12w) - (p + 12z)(b + 12y) \\ &= a(q + 12w) + 12xq' - (p + 12z)b - 12p'y \\ &= 1 + 12(aw + xq' - zb - p'y). \end{aligned}$$

Since $(p', q') = 1$, there exists integers x and y satisfying

$$p'y - q'x = aw - zb.$$

Then, the last term of (3.1) is equal to 1. Thus, we have $a'q' - b'p' = 1$, as required. \square

We denote by I_n the n -by- n identity matrix. We put

$$\Gamma = \{P \in SL(2, \mathbb{Z}) \mid P \equiv I_2 \pmod{12}\}.$$

Lemma 3.2. Γ is a normal closure of the set of the following 19 matrices.

$$\begin{aligned} P_0 &= \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}, & P_1 &= \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, & P_2 &= \begin{pmatrix} -143 & 12 \\ -12 & 1 \end{pmatrix}, \\ P_3 &= \begin{pmatrix} -155 & 84 \\ -24 & 13 \end{pmatrix}, & P_4 &= \begin{pmatrix} -191 & 156 \\ -60 & 49 \end{pmatrix}, & P_5 &= \begin{pmatrix} -443 & 120 \\ -48 & 13 \end{pmatrix}, \\ P_6 &= \begin{pmatrix} -467 & 360 \\ -48 & 37 \end{pmatrix}, & P_7 &= \begin{pmatrix} -299 & 108 \\ -36 & 13 \end{pmatrix}, & P_8 &= \begin{pmatrix} -311 & 216 \\ -36 & 25 \end{pmatrix}, \\ P_9 &= \begin{pmatrix} 937 & -396 \\ 168 & -71 \end{pmatrix}, & P_{10} &= \begin{pmatrix} 157 & -36 \\ 48 & -11 \end{pmatrix}, & P_{11} &= \begin{pmatrix} 157 & -48 \\ 36 & -11 \end{pmatrix}, \\ P_{12} &= \begin{pmatrix} 205 & -84 \\ 144 & -59 \end{pmatrix}, & P_{13} &= \begin{pmatrix} 157 & -72 \\ 24 & -11 \end{pmatrix}, & P_{14} &= \begin{pmatrix} 229 & -132 \\ 144 & -83 \end{pmatrix}, \\ P_{15} &= \begin{pmatrix} 169 & -108 \\ 36 & -23 \end{pmatrix}, & P_{16} &= \begin{pmatrix} 181 & -132 \\ 48 & -35 \end{pmatrix}, & P_{17} &= \begin{pmatrix} 589 & -108 \\ 60 & -11 \end{pmatrix}, \\ P_{18} &= \begin{pmatrix} 649 & -384 \\ 120 & -71 \end{pmatrix}. \end{aligned}$$

Proof. The GAP package Congruence [1] shows that Γ is generated by 97 matrices, as follows.

```
gap> LoadPackage("congruence");
true
gap> G:=PrincipalCongruenceSubgroup(12);
GeneratorsOfGroup(G); <principal congruence subgroup of level 12 in SL_2(Z)>
gap> GeneratorsOfGroup(G);
#I Using the Congruence package for GeneratorsOfGroup ...
[ [ [ 1, 12 ], [ 0, 1 ] ], [ [ -143, 12 ], [ -12, 1 ] ],
  [ [ 589, -108 ], [ 60, -11 ] ], [ [ 157, -36 ], [ 48, -11 ] ],
  [ [ -443, 120 ], [ -48, 13 ] ], [ [ 157, -48 ], [ 36, -11 ] ],
  [ [ -299, 108 ], [ -36, 13 ] ], [ [ 205, -84 ], [ 144, -59 ] ],
  [ [ 937, -396 ], [ 168, -71 ] ], [ [ 157, -72 ], [ 24, -11 ] ],
  [ [ -155, 84 ], [ -24, 13 ] ], [ [ 229, -132 ], [ 144, -83 ] ],
  [ [ 649, -384 ], [ 120, -71 ] ], [ [ 169, -108 ], [ 36, -23 ] ],
  [ [ -311, 216 ], [ -36, 25 ] ], [ [ 181, -132 ], [ 48, -35 ] ],
  [ [ -467, 360 ], [ -48, 37 ] ], [ [ -191, 156 ], [ -60, 49 ] ],
  [ [ 13, -12 ], [ 12, -11 ] ], [ [ 649, -768 ], [ 60, -71 ] ],
  [ [ 205, -252 ], [ 48, -59 ] ], [ [ -491, 624 ], [ -48, 61 ] ],
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[ [ 193, -252 ], [ 36, -47 ] ], [ [ -335, 456 ], [ -36, 49 ] ],
[ [ 349, -492 ], [ 144, -203 ] ], [ [ 181, -264 ], [ 24, -35 ] ],
[ [ -179, 276 ], [ -24, 37 ] ], [ [ 373, -588 ], [ 144, -227 ] ],
[ [ 205, -336 ], [ 36, -59 ] ], [ [ -347, 588 ], [ -36, 61 ] ],
[ [ 229, -396 ], [ 48, -83 ] ], [ [ -515, 912 ], [ -48, 85 ] ],
[ [ -251, 456 ], [ -60, 109 ] ], [ [ 25, -48 ], [ 12, -23 ] ],
[ [ 709, -1548 ], [ 60, -131 ] ], [ [ 253, -564 ], [ 48, -107 ] ],
[ [ -539, 1224 ], [ -48, 109 ] ], [ [ 229, -528 ], [ 36, -83 ] ],
[ [ -371, 876 ], [ -36, 85 ] ], [ [ 493, -1188 ], [ 144, -347 ] ],
[ [ 205, -504 ], [ 24, -59 ] ], [ [ -203, 516 ], [ -24, 61 ] ],
[ [ 517, -1332 ], [ 144, -371 ] ], [ [ 241, -636 ], [ 36, -95 ] ],
[ [ -383, 1032 ], [ -36, 97 ] ], [ [ 277, -756 ], [ 48, -131 ] ],
[ [ -563, 1560 ], [ -48, 133 ] ], [ [ -311, 876 ], [ -60, 169 ] ],
[ [ 37, -108 ], [ 12, -35 ] ], [ [ 301, -972 ], [ 48, -155 ] ],
[ [ 265, -876 ], [ 36, -119 ] ], [ [ -407, 1368 ], [ -36, 121 ] ],
[ [ 637, -2172 ], [ 144, -491 ] ], [ [ 229, -792 ], [ 24, -83 ] ],
[ [ -227, 804 ], [ -24, 85 ] ], [ [ 661, -2364 ], [ 144, -515 ] ],
[ [ 277, -1008 ], [ 36, -131 ] ], [ [ -419, 1548 ], [ -36, 133 ] ],
[ [ 325, -1212 ], [ 48, -179 ] ], [ [ -371, 1416 ], [ -60, 229 ] ],
[ [ 49, -192 ], [ 12, -47 ] ], [ [ 349, -1476 ], [ 48, -203 ] ],
[ [ 301, -1296 ], [ 36, -155 ] ], [ [ 781, -3444 ], [ 144, -635 ] ],
[ [ 253, -1128 ], [ 24, -107 ] ], [ [ -251, 1140 ], [ -24, 109 ] ],
[ [ 805, -3684 ], [ 144, -659 ] ], [ [ 313, -1452 ], [ 36, -167 ] ],
[ [ 373, -1764 ], [ 48, -227 ] ], [ [ -431, 2076 ], [ -60, 289 ] ],
[ [ 61, -300 ], [ 12, -59 ] ], [ [ 397, -2076 ], [ 48, -251 ] ],
[ [ 337, -1788 ], [ 36, -191 ] ], [ [ 277, -1512 ], [ 24, -131 ] ],
[ [ -275, 1524 ], [ -24, 133 ] ], [ [ 349, -1968 ], [ 36, -203 ] ],
[ [ 421, -2412 ], [ 48, -275 ] ], [ [ -491, 2856 ], [ -60, 349 ] ],
[ [ 73, -432 ], [ 12, -71 ] ], [ [ 445, -2772 ], [ 48, -299 ] ],
[ [ 373, -2352 ], [ 36, -227 ] ], [ [ 385, -2556 ], [ 36, -239 ] ],
[ [ 469, -3156 ], [ 48, -323 ] ], [ [ -551, 3756 ], [ -60, 409 ] ],
[ [ 85, -588 ], [ 12, -83 ] ], [ [ 493, -3564 ], [ 48, -347 ] ],
[ [ 409, -2988 ], [ 36, -263 ] ], [ [ 421, -3216 ], [ 36, -275 ] ],
[ [ 517, -3996 ], [ 48, -371 ] ], [ [ -611, 4776 ], [ -60, 469 ] ],
[ [ 97, -768 ], [ 12, -95 ] ], [ [ 541, -4452 ], [ 48, -395 ] ],
[ [ 565, -4932 ], [ 48, -419 ] ], [ [ -671, 5916 ], [ -60, 529 ] ],
[ [ 109, -972 ], [ 12, -107 ] ], [ [ 121, -1200 ], [ 12, -119 ] ],
[ [ 133, -1452 ], [ 12, -131 ] ] ]
```

We can verify that the set of the above 97 matrices is equal to

$$\{P_0\} \cup \{T^j P_1 T^{-j} | j = 1, 2, \dots, 11\} \cup \bigcup_{n=2}^{18} \{T^j P_n T^{-j} | j = 0, 1, \dots, m_n\},$$

where we put

$$\begin{aligned} m_2 &= 0, \quad m_3 = 5, \quad m_4 = 8, \quad m_5 = 2, \quad m_6 = 2, \quad m_7 = 3, \\ m_8 &= 3, \quad m_9 = 0, \quad m_{10} = 8, \quad m_{11} = 7, \quad m_{12} = 4, \quad m_{13} = 5, \\ m_{14} &= 4, \quad m_{15} = 7, \quad m_{16} = 8, \quad m_{17} = 2, \quad m_{18} = 0, \end{aligned}$$

completing the proof. \square

Lemma 3.3. $\Gamma \subset \ker \rho$, that is, $\rho(P) = I_{10}$ for any $P \in \Gamma$.

Proof. From Lemma 3.2, it is sufficient to show that $\rho(P_n) = I_{10}$ for $n = 0, 1, \dots, 18$. We can verify that these matrices is presented as the products of S and T , as follows.

$$\begin{aligned} P_0 &= T^{12}, & P_1 &= S^3 T^{-12} S, \\ P_2 &= S^2 T^{12} S T^{12} S, & P_3 &= S^2 T^7 S T^2 S T^7 S T^2 S, \\ P_4 &= S^2 T^3 S T^{-5} S T^2 S T^{-4} S T S, & P_5 &= T^9 S T^{-4} S T^3 S T^4 S, \\ P_6 &= T^{10} S T^4 S T^3 S T^{-3} S T S, & P_7 &= T^8 S T^{-3} S T^4 S T^3 S, \\ P_8 &= T^9 S T^3 S T^4 S T^{-2} S T S, & P_9 &= T^5 S T^{-2} S T^{-4} S T^{-4} S T^{-3} S T^2 S, \\ P_{10} &= T^3 S T^{-4} S T^{-3} S T^4 S, & P_{11} &= T^4 S T^{-3} S T^{-4} S T^3 S, \\ P_{12} &= T S T^{-2} S T^3 S T^4 S T^{-2} S T^2 S, & P_{13} &= T^6 S T^{-2} S T^{-6} S T^2 S, \\ P_{14} &= T S T^{-2} S T^{-3} S T^4 S T^4 S T^2 S, & P_{15} &= S^2 T^5 S T^3 S T^{-3} S T^2 S T^2 S, \\ P_{16} &= T^4 S T^4 S T^{-3} S T^{-3} S T S, & P_{17} &= S^2 T^{10} S T^5 S T^{-2} S T^5 S, \\ P_{18} &= T^5 S T^{-2} S T^2 S T^{-4} S T^3 S T^2 S. \end{aligned}$$

By using these formulae, we can verify that ρ takes to each of the matrices to I_{10} , completing the proof of the lemma. \square

Proposition 3.4. Let p, q, p', q' be integers satisfying $(p, q) = 1$, $(p', q') = 1$ and $p \equiv p', q \equiv q' \pmod{12}$. Then, $Z(L(p, q)) = Z(L(p', q'))$.

Proof. From Lemma 3.1, there exist matrices

$$A = \begin{pmatrix} -q & b \\ p & -a \end{pmatrix}, \quad A' = \begin{pmatrix} -q' & b' \\ p' & -a' \end{pmatrix} \in SL(2, \mathbb{Z})$$

such that $A \equiv A' \pmod{12}$. We put $P = I_2 + A^{-1}(A' - A)$. By definition, $P \in \Gamma$ and $A' = AP$. Thus, by (2.1) and Lemma 3.3, we have

$$Z(L(p', q')) = w^t \mathbf{e} \rho(A') \mathbf{e} = w^t \mathbf{e} \rho(A) \rho(P) \mathbf{e} = w^t \mathbf{e} \rho(A) \mathbf{e} = Z(L(p, q)),$$

completing the proof. \square

Proof of Theorem 1.1. By Proposition 3.4, the left-hand side of (1.1) has period 12 for p and q . On the other hand, the right-hand side of (1.1) also has period 12 for p and q . By [4, Appendix D], we can verify that (1.1) holds for coprime integers p and q with $1 \leq p \leq 12$, $q < p$. Therefore, (1.1) holds for any coprime integers p and q . \square

Proof of Corollary 1.2. It is known, see [3, Remark 3], that there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p, q)$ and $L(p', q')$ if and only if $p = p'$ and $q \equiv n^2 q' \pmod{p}$ for some integer n .

When $(p, 12) \neq 3, 4, 12$, by Theorem 1.1 the value $Z(L(p, q))$ does not depend on q . Thus, $Z(L(p, q)) = Z(L(p', q'))$.

When $(p, 12) = k$ with $k \in \{3, 4, 12\}$, we have $q \equiv n^2 q' \pmod{k}$ for some integer n . Thus, it is enough to show that $q \equiv q' \pmod{k}$.

- (a) When $k \in \{3, 4\}$, we have $\{n^2 | n \in \mathbb{Z}/k\mathbb{Z}\} = \{0, 1\}$. Since $(p, q) = 1$, it implies that $q \equiv q' \pmod{k}$.
- (b) When $k = 12$, we have $\{n^2 | n \in \mathbb{Z}/12\mathbb{Z}\} = \{0, 1, 4, 9\}$. Again, since $(p, q) = 1$, it implies that $q \equiv q' \pmod{k}$.

Thus, by Theorem 1.1, $Z(L(p, q)) = Z(L(p', q'))$. \square

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